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A criterion for the existence of four limit cycles in quadratic systems☆

G.A. Leonov

St Petersburg Russia, Russian Federation

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ABSTRACT

A method for the asymptotic integration of the trajectories is proposed for the Liénard equation. The results obtained by this method are used to prove the existence of two "large" limit cycles in quadratic systems with a weak focus. The application of standard procedures of small perturbations of the parameters of quadratic systems enables one to find additionally two "small" limit cycles. It is shown that the criterion obtained for the existence of four limit cycles generalizes the well known Shi theorem.

1. Introduction

The determination of the number of limit cycles [1] of the quadratic system

$$\begin{aligned}\dot{x} &= a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y\\ \dot{y} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y\end{aligned} \tag{1.1}$$

is an important part of Hilbert's sixteenth problem.[2-10]

It is well known that, for almost all values of the parameters, system (1.1) can be transformed to the Liénard equation [11-16]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.2}$$

with the functions

$$\begin{aligned}f(x) &= (Ax^2 + Bx + C)|x+1|^{q-2}\\ g(x) &= (C_1x^3+C_2x^2+C_3x+C_4)x\frac{|x+1|^{2q}}{(x+1)^3}\end{aligned}$$

where A, B, C, C_j and q are certain numbers.

The case of degeneracy of the zero singular point, when $C=0$ and $C_4=1$ is of particular interest. We shall henceforth assume that this condition is satisfied.

For the existence of an inverse transformation from Eq. (1.2) to system (1.1), it is necessary and sufficient that the relations [16]

$$\begin{aligned}&\frac{B-A}{(2q-1)^2}((1-q)B + (3q-2)A) = 2C_2 - 3C_1 - C_3\\ &\frac{B-A}{(2q-1)^2}\left(B + 2(q-1)A\right) = C_2 - 2C_1 - 1\end{aligned} \tag{1.3}$$

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E-mail address: leonov@math.spbu.ru.

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are satisfied. In this case, the parameters of system (1.1) are calculated as follows:

$$a_1 = 1 + \frac{B-A}{2q-1}, \quad a_2 = -(q+1)a_1^2 - Aa_1 - C_1$$
$$b_1 = \beta_1 = \alpha_1 = 1, \quad b_2 = -A - a_1(2q+1), \quad \alpha_2 = -2, \quad \beta_2 = -1, \quad c_1 = 0, \quad c_2 = -q$$

The problem of determining the number of limit cycles of system (1.1) can therefore be reformulated as the problem of determining the number of limit cycles of Eq. (1.2) when relations (1.3) are satisfied.

The Liénard equation (1.2) describes the oscillations of many mechanical, electrical and electronic systems and the investigation of the limit cycles of this equation has its own long history.[17-20]

In the spirit of developing these investigations, a method for the asymptotic integration of the trajectories of Eq. (1.2) is proposed in this paper and, using this, a criterion for the existence of two "large" limit cycles is established. At the same time, two small limit cycles in the neighbourhood of the degenerate focus can be obtained using small perturbations of the parameters of system (1.1).

2. Reduction of the Liénard equation to a special form

We shall henceforth assume that

$$B = Ab, \ b > 1, \ A < 0, \ q \in (-1,0)$$

and that the first Lyapunov quantity of the zero equilibrium state of Eq. (1.2) is equal to zero:

$$f''(0) = f'(0)g''(0) \tag{2.1}$$

In this case, when account is taken of equalities (1.3) and (2.1), the functions $f(x)$ and $g(x)$ in Eq. (1.2) can be written in the following form

$$f(x) = \tilde{f}(x)|x+1|^{q-2}, \ \tilde{f}(x) = A\Big((x+1)^2 + (b-2)(x+1) + (1-b)\Big)$$
$$g(x) = \tilde{g}(x)\frac{|x+1|^{2q}}{(x+1)^3}, \ \tilde{g}(x) = \Big(D_1(x+1)^4 + D_2(x+1)^3 + D_3(x+1)^2 + D_4(x+1) + D_5\Big)$$

where

$$D_1 = \frac{-1}{b(2q-1)^2}\Big((b-1)b(b+bq+q-2)A^2 + (2q-1)^2(bq-1+b)\Big)$$
$$D_2 = \frac{1}{b(2q-1)^2}\Big((b-1)b(3b+2bq-6+4q)A^2 + (2q-1)^2(2bq-2+3b)\Big)$$
$$D_3 = \frac{-1}{b(2q-1)^2}\Big(3(b-1)b(b+2q-2)A^2 + (2q-1)^2(bq-1+2b)\Big)$$
$$D_4 = \frac{-(b-1)(b-2)A^2}{(2q-1)}, \ D_5 = \frac{q(b-1)^2A^2}{(2q-1)^2}$$

Equation (1.2) is equivalent to the first order equation

$$F\frac{dF}{dx} + f(x)F + g(x) = 0$$

which can be written in the following forms

$$FdF + \frac{\tilde{f}(x)}{q+1}(x+1)^{-2}Fd(x+1)^{q+1} + \frac{\tilde{g}(x)}{q+1}(x+1)^{q-3}d(x+1)^{q+1} = 0, \ x \geq 0 \tag{2.2}$$

$$FdF + \frac{\tilde{f}(x)}{q-1}Fd(x+1)^{q-1} + \frac{\tilde{g}(x)}{q-1}(x+1)^{q-1}d(x+1)^{q-1} = 0, \ x \in (-1,0) \tag{2.3}$$

For Eq. (2.2), we make the substitution $z = (x+1)^{q+1}$, and, for Eq. (2.3), the substitution $z=(x+1)^{q-1}$. In this case, Eqs (2.2) and (2.3) are written as follows

$$FdF + \frac{A}{q+1}\xi^+(z)Fdz + \frac{1}{q+1}\eta^+(z)zdz = 0, z \geq 1 \tag{2.4}$$

$$FdF + \frac{A}{q-1}\xi^-(z)z^{\frac{2}{q-1}}Fdz + \frac{1}{q-1}\eta^-(z)z^{1+\frac{4}{q-1}}dz = 0, z \geq 1 \tag{2.5}$$

Here,

$$\xi^{\pm}(z) = 1 + (b-2)z^{-\frac{1}{q\pm1}} + (1-b)z^{-\frac{2}{q\pm1}},$$

$$\eta^{\pm}(z) = D_1 + D_2 z^{-\frac{1}{q\pm1}} + D_3 z^{-\frac{2}{q\pm1}} + D_4 z^{-\frac{3}{q\pm1}} + D_5 z^{-\frac{4}{q\pm1}}$$

The form of Eqs (2.4) and (2.5) is "well adapted" to asymptotic analysis of trajectories with large initial data since the terms $z^{k/q+1}$ and $z^{k/q-1} (k = 1, \ldots, 4)$ are infinitesimal "at infinity", and, by discarding them, it is possible to transfer to an analysis of second order equations with constant coefficients.

The scheme described here will be realized in the following section.

3. The method of asymptotic integration of the trajectories of the Liénard equation

We fix a certain number $\delta > 0$ and introduce a basily large number R into the treatment.Assuming that the condition

$$4D_1(q+1) > A^2 \tag{3.1}$$

is satisfied, we introduce the notation

$$\lambda = -\frac{A}{2(q+1)}, \quad \omega = \frac{\sqrt{4D_1(q+1) - A^2}}{2(q+1)}$$

The following result holds for the system

$$\dot{x} = y,\ \dot{y} = -f(x)y - g(x) \tag{3.2}$$

Lemma 1. For the solution of system (3.2) with the initial conditions $x(0) = 0$, $y(0) = R$, a number $T > 0$ exists such that

$$x(T) = 0,\ y(T) < 0,\ x(t) > 0,\ \forall t \in (0,T)$$
$$R\exp\left(\frac{\lambda\pi}{\omega} - \delta\right) < |\,y(T)\,| < R\exp\left(\frac{\lambda\pi}{\omega} + \delta\right)$$

Proof *(.)*. System (3.2) is equivalent to Eq. (2.4) with $F(1) = R$. Note that, for any (as small as desired) $\varepsilon > 0$, a number Z exists such that

$$1 - \varepsilon < \xi^{+}(z) < 1 + \varepsilon,\ D_1 - \varepsilon < \eta^{+}(z) < D_1 + \varepsilon,\ \forall z \geq Z \tag{3.3}$$

and, in the interval $[1,Z]$ for large R, the solution $F(z)$ considered is close to the solution $\tilde{F}(z)$ of the equation

$$\frac{d\tilde{F}}{dz} + \frac{A}{q+1}\xi^{+}(z) = 0, \quad \tilde{F}(1) = R$$

Hence, in the interval $[1,Z]$,

$$F(z) = R - \int_1^z \frac{A}{q+1}\xi^{+}(s)ds + \kappa(R);\ \kappa(R) \to 0 \text{ при } R \to +\infty$$

It follows from this that

$$(1 - \kappa_1(R))R \leq F(Z) \leq (1 + \kappa_1(R))R;\ \kappa_1(R) \to 0 \text{ when } R \to +\infty \tag{3.4}$$

We introduce the notation

$$L(F;\varepsilon) = F\frac{dF}{dz} + \frac{A}{q+1}(1+\varepsilon)F + \frac{1}{q+1}(D_1 - \varepsilon)z$$

It is easily seen that the estimate $F(z) \leq F_1(z)$ holds when $z \geq Z$, where $F_1(z)$ is the solution of the equation

$$L(F_1;\varepsilon) = 0,\ F_1(Z) = F(Z)$$

To do this, it is necessary to consider the curve $F = F_1(z)$ and the relation on this curve

$$\frac{dF_1}{dz} - \frac{dF}{dz} > 0, \quad \forall z \geq Z$$

In a similar manner, we obtain the estimate $F(z) \geq F_2(z)$, where $F_2(z)$ is the solution of the equation

$$L(F_2;-\varepsilon) = 0,\ F_2(Z) = F(Z)$$

The relative position of the solutions is shown in Fig. 1. The vectors corresponding to the values of dF/dz and to the vector field of system (3.2) are indicated by the arrows.

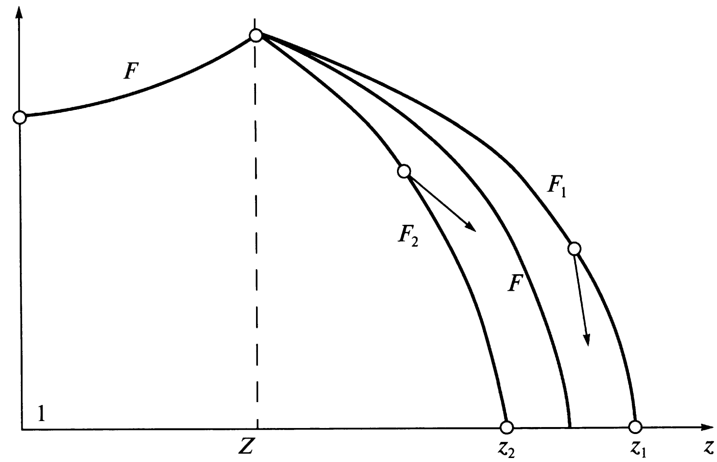

Fig. 1.

Since ε is an arbitrarily small number, the values of $F = F_1(z)$ and $F = F_2(z)$ are close to the values of the derivative $\dot{z}(t)$ of the solution $z(t)$ of the equation

$$\ddot{z} + \frac{A}{q+1}\dot{z} + \frac{D_1}{q+1}z = 0,\ z(0) = Z,\ \dot{z}(0) = F(Z) \tag{3.5}$$

which we write in the form

$$z(t) = (c_1 \sin\omega t + c_2 \cos\omega t)e^{\lambda t};\quad c_1 = \frac{1}{\omega}(F(Z) - \lambda Z),\quad c_2 = Z,\quad \lambda = -\frac{A}{2(q+1)}$$

Similar reasoning can be used for the lower half-plane $\{F \leq 0, z \geq 1\}$ (Fig. 2).Here, $F_4(z) \leq F(z) \leq F_3(z)$, where $F_3(z)$ and $F_4(z)$ are the negative solutions of the equations

$$L(F_3; -\varepsilon) = 0,\ F_3(z_2) = 0$$

$$L(F_4; -\varepsilon) = 0,\ F_4(z_1) = 0$$

respectively, and z_1 and z_2 are zeroes of the functions $F_1(z)$ and $F_2(z)$ respectively.

Note that, in the case of small ε, the values of $F = F_3(z)$ and $F = F_4(z)$ are also close to the values of the derivative $\dot{z}(t)$ of the solution $z(t)$ of Eq. (3.5).

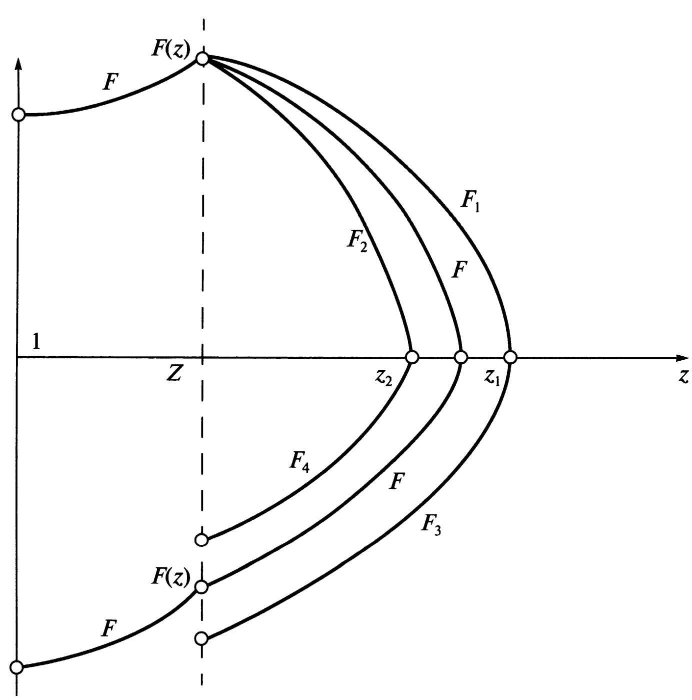

Fig. 2.

We recall that the relation $\varepsilon(Z)\to 0$ when $Z\to+\infty$ holds in the equations obtained above for the parameter ε.

The estimate

$$z(T)=Z,\quad \dot z(T)<0$$

holds for the time of intersection of the solution $z(t)$, $\dot z(t)$ with the line $\{z=Z\}$ when $t>0$

$$T=\frac{\pi}{\omega}+\kappa_2(R),\ \kappa_2(R)\to 0 \text{ when } R\to+\infty$$

But, then,

$$-F(Z)\exp\left(\frac{\lambda\pi}{\omega}+\kappa_3(R)\right)\le \dot z(T)\le -F(Z)\exp\left(\frac{\lambda\pi}{\omega}-\kappa_3(R)\right);\ \kappa_3(R)\to 0 \text{ when } R\to+\infty \tag{3.6}$$

Here, $F(Z)>0$.

Note that the estimate

$$F(Z)-\dot z(T)=O(\varepsilon)$$

holds for $F(Z)<0$ (Fig. 2).

Hence, when $Z\ll R$, we obtain the assertion of Lemma 1 from the relation $\varepsilon(Z)\to 0$ when $Z\to+\infty$ and from the estimates (3.4) and (3.6). In this case, the estimation of the negative value of $F(Z)$ in the interval $[1,Z]$ is analogous to the estimation of the positive value of $F(Z)$ in this interval carried out above.

We now introduce a certain number $c<-1$ into the treatment.

Lemma 2. Suppose condition (3.1) is satisfied. A number T > 0 exists for the solution of system (3.2) with initial conditions $x(0)=c$, $y(0)=-R$ such that

$$x(T)=c,\ y(T)>0,\ x(t)<c,\ \forall t\in(0,T)$$

$$R\exp\left(\frac{\lambda\pi}{\omega}-\delta\right)<y(T)<R\exp\left(\frac{\lambda\pi}{\omega}+\delta\right)$$

The proof of Lemma 2 is analogous to the proof of Lemma 1.

Lemma 3. Suppose the condition

$$A(1-b)>0,\ A^2(1-b)^2>4D_5(q-1)>0 \tag{3.7}$$

is satisfied. Then, a number $T>0$ exists for the solution of system (3.2) with initial conditions $x(0)=0$, $y(0)=-R$ such that

$$x(T)=0,\ 0<y(T)<\delta R,\quad x(t)\in(-1,0),\ \forall t\in(0,T)$$

Proof *(.)*. Note that system (3.2) is equivalent to Eq. (2.5) with $z\ge 1$ and $F(1)=-R$.

We will now consider this equivalence relation in greater detail.

The transformation

$$U: z=(x+1)^{q-1},\ x\in(-1,0]$$

maps the trajectory $x(t)$, $y(t)$ of system (3.2) (or the solution $F(x)$) onto the solution $F(z)$, as shown in Fig. 3.

When account is taken of the transformation U, the scheme of the proof basically repeats the technique developed for the proof of Lemma 1.

Here, for any $\varepsilon>0$, a number z exists such that

$$\frac{A(1-b)}{q-1}-\varepsilon<\frac{A}{q-1}\xi^-(z)z^{\frac{2}{q-1}}<\frac{A(1-b)}{q-1}+\varepsilon,\quad \forall z\ge Z \tag{3.8}$$

$$\frac{D_5}{q-1}-\varepsilon<\frac{1}{q-1}\eta^-(z)z^{\frac{4}{q-1}}<\frac{D_5}{q-1}+\varepsilon,\quad \forall z\ge Z \tag{3.9}$$

For large values of R, the solution $F(z)$ $(F(1)=-R)$ is close to the solution of the equation

$$\frac{d\tilde F}{dz}+\frac{A}{q-1}\xi^-(z)z^{\frac{2}{q-1}}=0,\ \tilde F(1)=-R$$

Hence, in $[1,Z]$, we obtain the estimate

$$(-1-\kappa_1(R))R\le F(Z)\le(-1+\kappa_1(R))R;\ \kappa_1(R)\to 0 \text{ when } R\to+\infty$$

which is analogous to the estimate (3.4).

As in the proof of Lemma 1, we obtain the estimates (Fig. 4)

$$F_1(z)\le F(z)\le F_2(z),\ F_2(Z)=F(Z)=F_1(Z)$$

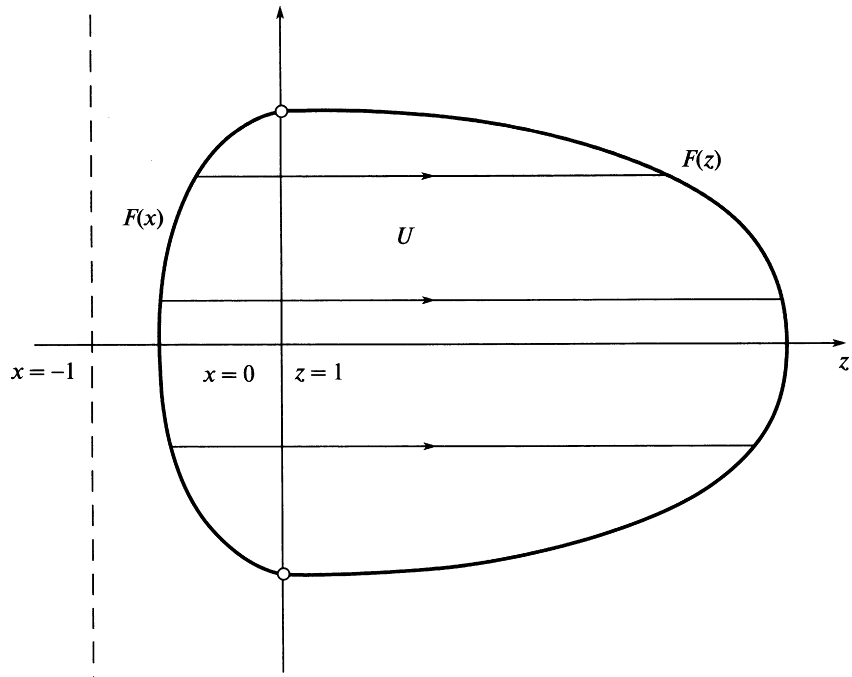

Fig. 3.

from relations (3.8) and (3.9), where $F_1(z)$ is the solution of the equation

$$M(F_1;\varepsilon)=0;\quad M(F_1;\varepsilon)=F_1\frac{dF_1}{dz}+\left(\frac{A(1-b)}{q-1}-\varepsilon\right)F_1+\left(\frac{D_5}{q-1}+\varepsilon\right)z$$

and $F_2(z)$ is the solution of the equation

$$M(F_2;-\varepsilon)=0$$

Since ε is an arbitrarily small number, the values of F_1 and F_2 are close to the values of the derivative $\dot{z}(t)$ of the solution of the equation

$$\ddot{z}+\frac{A(1-b)}{q-1}\dot{z}+\frac{D_5}{q-1}z=0,\ z(0)=Z,\ \dot{z}(0)=F(Z) \tag{3.10}$$

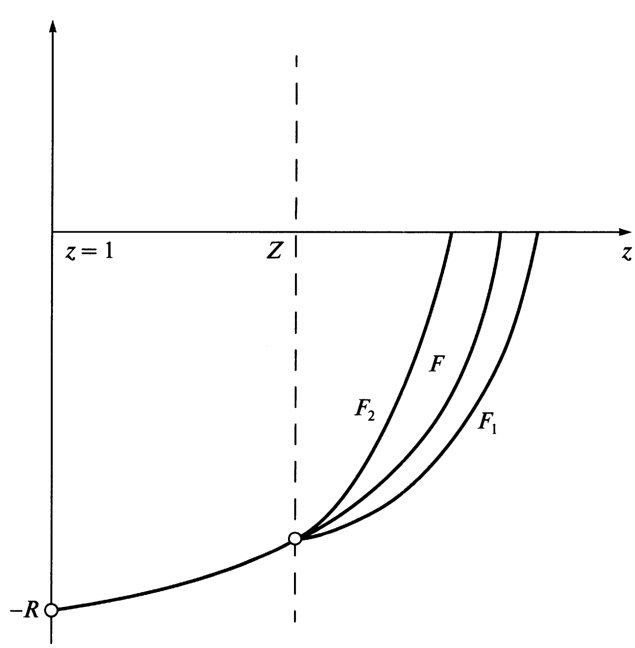

Fig. 4.

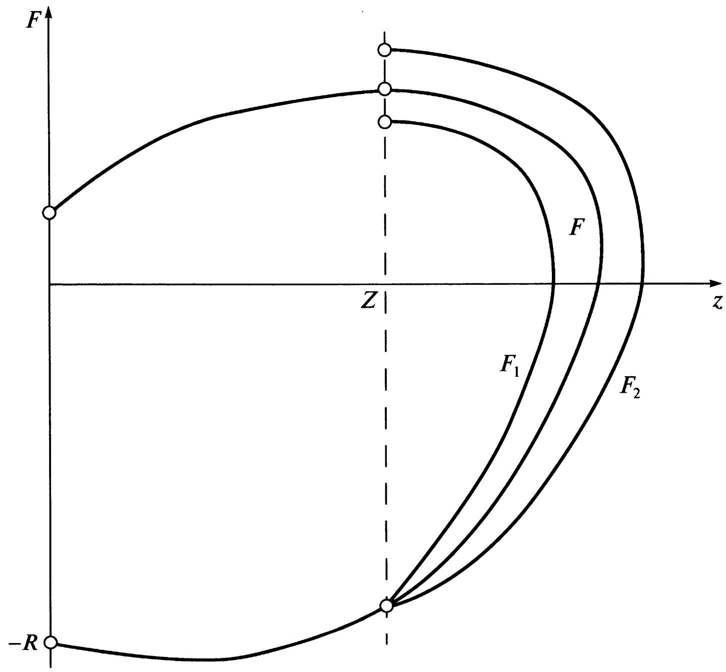

Fig. 5.

which, when condition (3.7) is taken into account, can be written in the form (here, $t<0$)

$$z(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$c_1 = \frac{F(Z) - \lambda_2 Z}{\lambda_1 - \lambda_2}, \quad c_2 = \frac{F(Z) - \lambda_1 Z}{\lambda_2 - \lambda_1}, \quad \lambda_{1,2} = -\frac{A(1-b)}{2(q-1)} \pm \sqrt{\frac{A^2(1-b)^2}{4(q-1)^2} - \frac{D_5}{q-1}}$$

Similar reasoning can also be carried through for the upper half plane $\{F>0, z\geq 0\}$ (Fig. 5), where the derivative $\dot{z}(t)$ of the solution $z(t)$ of Eq. (2.15) is also found to be close to $F(z)$. In this case, for the time of intersection with the line $\{z=Z\}$:$z(T)=Z(T<0)$, we have the estimate

$$\dot{z}(T) < R\varkappa_2(R), \varkappa_2(R) \to 0 \text{ when } R \to +\infty$$

The latter follows from the positiveness of λ_1 and λ_2.

The assertion of the lemma follows from this.

An assertion, analogous to the assertion of Lemma 3, also holds for the case when $x<-1$.

4. A Criterion for the existence of limit cycles

From the results of the preceding section we obtain the following theorem.

Theorem 1. Suppose the conditions $A<0$, $b>1$, $q\in(-1,0)$, (3.1) and (3.7) are satisfied. Then, the trajectories of system (3.2) for sufficiently large initial data:

$$|x(0)| + |y(0)| \gg 1, \quad x(0) \neq -1$$

will behave as shown in Fig. 6.

Note that relation (3.7) is satisfied for all values of $A<0$, $b>1$, $q\in(-1, 0)$.

Condition (3.1) can be written in the form

$$A^2 < \frac{(q+1)(2q-1)^2(1-b(q+1))}{b((q+1)b-3/2)^2}. \tag{4.1}$$

Hence, if condition (4.1) is satisfied, then, according to Theorem 1, the arrangement of the trajectories will be as shown in Fig. 6.

As was stated at the beginning of Section 2, the first Lyapunov quantity at the zero point is equal to zero: $L_1(0)=0$.

The second Lyapunov quantity at zero has the following expression:

$$L_2(0) = -\frac{\pi A(b-1)(4b+2bq-5)\Lambda}{24b}, \quad \Lambda = \frac{A^2 b(b(q+1)-2) + (q+1)(2q-1)^2}{(2q-1)^2}$$

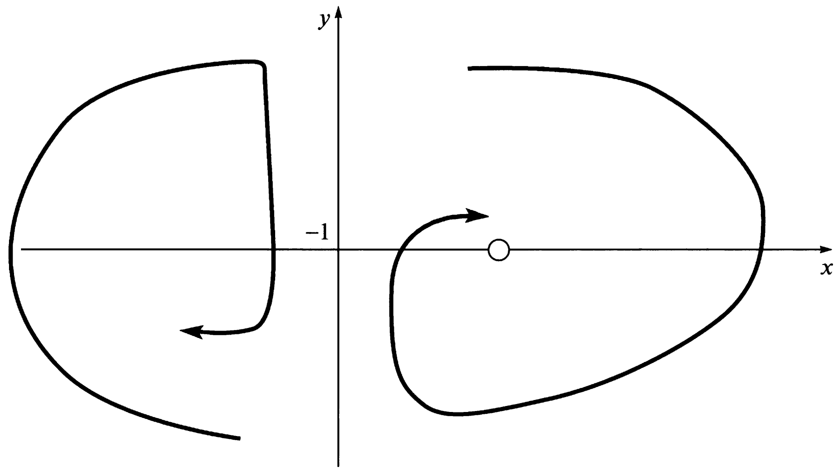

Fig. 6.

We will assume that $g(x)$ has only one zero $x=x_1$ in $(-\infty, -1)$. The inequality

$$g(-b) > 0;\ g(-b) = b^2(b-1)^{2q-1}\Lambda$$

is then the condition that $f(x_1) < 0$.

Since it is necessary that $b(q+1) < 1$ in the inequality (4.1), it is easily seen that, for $L_2(0)$ and $g(-b)$ to be positive, it is sufficient that inequality (4.1) is satisfied and

$$5/2 - b < b(q+1) < 1 \tag{4.2}$$

From this and Theorem 1 we obtain the following theorem.

Theorem 2. Suppose the function g(x) has unique zeroes in the intervals $(-\infty, -1)$ and $(-1,+-\infty)$, and also $A < 0$, $b > 1$, $q \in (-1,0)$ and inequality (4.1) is satisfied.

System (3.2) then has a limit cycle located in the half-plane $\{x < -1, y \in R^1\}$.

If, in addition to this, condition (4.2) is satisfied, system (3.2) has two limit cycles located in the domains $\{x < -1, y \in R^1\}$ and $\{x > -1, y \in R^1\}$.

We observe that, when the equality

$$2b(q+2) = 5 \tag{4.3}$$

and the conditions of Theorem 2 are satisfied, the second Lyapunov quantity L_2 (0) is equal to zero and the inequality $L_3(0) < 0$ holds for the third Lyapunov quantity.

It is well known [4-6] that, in the case when $L_1(0) = 0$, $L_2(0) \neq 0$, classes of systems with two small limit cycles in the neighbourhood of the point $x = y = 0$ can be separated out by small perturbations of the parameters of quadratic systems and, consequently, also, of system (3.2). If $L_1(0) = L_2(0) = 0$ and $L_3(0) \neq 0$, then a similar procedure separates out three small limit cycles in the neighbourhood of the zero equilibrium state.

Hence, the conditions formulated in Theorem 2 plus the above mentioned perturbation of the parameters separates out the classes of systems (1.1) and (1.2) with four limit cycles.

The case of (4.3) has been intensively studied in a large number of papers. The article [21] was one of the first papers where four limit cycles were revealed in this case.

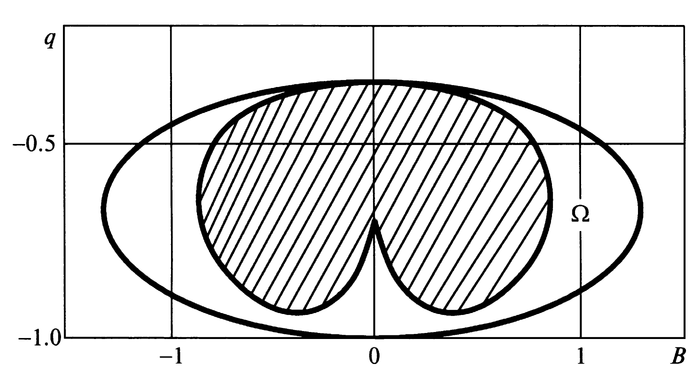

Fig. 7.

In the case of condition (4.3), relation (4.1) takes the form

$$B^2 < -5(q+1)(3q+1) \tag{4.4}$$

where $B = Ab$. The domain Ω, where inequality (4.4) is satisfied, is presented in Fig. 7 in the plane of the parameters B and q. The hatched region (it is wholly contained in the domain Ω) is the set of parameters (B, q) where four cycles exist (after the above mentioned small perturbations of the parameters) which was obtained earlier. [21]

Shi's well known theorem [21] is therefore generalized here.

Note also that Theorem 2 distinguishes a three-dimensional domain in the space of the parameters A, b and q corresponding to four limit cycles, where equality (4.3) only defines a certain section of this domain.

The problem of organizing numerical procedures to search for limit cycles in the domain of the parameters A, b and q, distinguished by the conditions of Theorem 2, naturally arises. This work has been started in Ref. 22.

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